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A non-reductive $N = 4$ superconformal algebra

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Abstract

A new $N = 4$ superconformal algebra (SCA) is presented. Its internal affine Lie algebra is based on the seven-dimensional Lie algebra $su(2) \oplus g$, where g should be identified with a four-dimensional non-reductive Lie algebra. Thus, it is the first known example of what we choose to call a non-reductive SCA. It contains a total of 16 generators and is obtained by a non-trivial Inönü–Wigner contraction of the well-known large $N = 4$ SCA. The recently discovered asymmetric $N = 4$ SCA is a subalgebra of this new SCA. Finally, the possible affine extensions of the non-reductive Lie algebra g are classified. The two-form governing the extension appearing in the SCA differs from the ordinary Cartan–Killing form.

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1. Introduction

$N = 4$ superconformal algebras (SCAs) in two dimensions have been studied extensively [1–5]. Their internal affine Lie algebras are all Kac–Moody (KM) algebras based on reductive Lie algebras. We recall that a reductive Lie algebra g admits a decomposition into a semi-simple Lie algebra and a direct sum of $u(1)$'s:

$$g = g_1 \oplus \cdots \oplus g_n \oplus \underbrace{u(1) \oplus \cdots \oplus u(1)}_m \quad (1)$$

where g_i is a simple Lie algebra, and n or m may be zero. To the best of our knowledge, all known SCAs have an internal KM algebra based on a reductive Lie algebra. In the simple case of only one supercurrent, $N = 1$, the internal symmetry group is generated by the identity, i.e. $n = m = 0$.

The internal affine Lie algebra of the small $N = 4$ SCA [1] is based on $su(2)$; the large $N = 4$ SCA [2, 3] on $su(2) \oplus su(2) \oplus u(1)$; the middle $N = 4$ SCA [4] on $su(2) \oplus u(1) \oplus u(1) \oplus u(1) \oplus u(1)$; while the asymmetric $N = 4$ SCA [5] is based on $su(2) \oplus u(1) \oplus u(1)$. The total number of generators in the four examples listed above is 8,

16, 16 and 12, respectively. Besides the Virasoro generator, the four supercurrents, and the affine Lie algebra generators, the remaining generators are all spin-1/2 fields.

It is possible to extend the asymmetric $N = 4$ SCA by considering one of the $u(1)$ generators as the derivative of a scalar. It is this slightly bigger SCA that is provided in [5].

Here we shall consider a non-trivial Inönü–Wigner (IW) contraction of the large $N = 4$ SCA. The resulting algebra is of course closed under (anti-)commutators, whereas the Jacobi identities are not necessarily satisfied. However, we have checked explicitly that they are. They read

$$0 = (-1)^{ac}[A_r, [B_s, C_t]] + (-1)^{ba}[B_s, [C_t, A_r]] + (-1)^{cb}[C_t, [A_r, B_s]] \quad (2)$$

where a is the parity of the field A , etc. $[\cdot, \cdot]$ denotes an anti-commutator when both generators are fermionic, i.e., of odd parity. It is otherwise a commutator.

The internal symmetry group of the resulting $N = 4$ SCA turns out to be generated by an affine Lie algebra based on the seven-dimensional Lie algebra $su(2) \oplus g$, where g is a four-dimensional *non-reductive* Lie algebra to be discussed below. This novel kind of $N = 4$ SCA has 16 generators: the Virasoro generator, $N = 4$ supercurrents, the seven-dimensional affine Lie algebra and four spin-1/2 fermions.

Affine extensions of non-reductive Lie algebras are in general not unique. Thus, the affine extension of g that appears in our $N = 4$ SCA is dictated by the complex structure of the full SCA. We classify the possible affine extensions of g , and find that the extension relevant to our construction differs from the more conventional one governed by the ordinary Cartan–Killing form.

The guiding principle while considering the particular and somewhat peculiar IW contraction alluded to above, was to look for an extension of the recently discovered asymmetric $N = 4$ SCA [5]. Indeed, the latter appears as a subalgebra of the likewise asymmetric new and non-reductive $N = 4$ SCA, rendering its unfamiliar number of 12 generators less mysterious.

In the string theoretical context of the *AdS/CFT* correspondence, some explicit constructions of superconformal algebras have been obtained. The Virasoro algebra was considered in [6], $N = 1, 2$ and 4 SCAs were discussed in [7], while a general approach to constructing SCAs on the boundary of *AdS*₃ was outlined in [5]. Those works rely on free field realizations of affine current (super-)algebras on the world sheet of the string theory. It would be interesting to see how the new $N = 4$ SCA presented here fits into that framework. Free field realizations of generic affine current superalgebras first appeared in [8].

2. Large $N = 4$ superconformal algebra

Following [3], the large $N = 4$ SCA is generated by L , G^a , $A^{\pm i}$, U and Q^a with $a = 1, 2, 3, 4$ and $i = 1, 2, 3$. The conformal weights Δ for $\{G, A, U, Q\}$ are $\{3/2, 1, 1, 1/2\}$. A and U generate an affine $su(2) \oplus su(2) \oplus u(1)$ Lie algebra. The large $N = 4$ SCA is a two-parameter (doubly extended) algebra in terms of c and γ or equivalently k^+ and k^- :

$$\begin{aligned} k^+ &= \frac{c}{6\gamma} & k^- &= \frac{c}{6(1-\gamma)} \\ c &= \frac{6k^+k^-}{k^+ + k^-} & \gamma &= \frac{k^-}{k^+ + k^-}. \end{aligned} \quad (3)$$

For convenience of notation, one introduces 4×4 -matrices α satisfying

$$[\alpha^{\pm i}, \alpha^{\pm j}] = - \sum_{k=1}^3 \epsilon^{ijk} \alpha^{\pm k} \quad [\alpha^{+i}, \alpha^{-j}] = 0 \quad \{\alpha^{\pm i}, \alpha^{\pm j}\} = -\frac{1}{2} \delta^{ij}. \quad (4)$$

They can be represented by

$$\alpha_{ab}^{\pm i} = \pm \frac{1}{2} (\delta_a^i \delta_b^4 - \delta_b^i \delta_a^4) + \frac{1}{2} \epsilon^{iab}. \tag{5}$$

The large $N = 4$ SCA may now be written

$$\begin{aligned} [L_n, L_m] &= (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \\ [L_n, \Phi_r] &= ((\Delta(\Phi) - 1)n - r)\Phi_{n+r} \\ \{G_r^a, G_s^b\} &= 2\delta^{ab}L_{r+s} - 4(r - s) \sum_{i=1}^3 (\gamma\alpha_{ab}^{+i}A_{r+s}^{+i} + (1 - \gamma)\alpha_{ab}^{-i}A_{r+s}^{-i}) + \frac{c}{3}(r^2 - 1/4)\delta^{ab}\delta_{r+s,0} \\ [A_n^{+i}, G_r^a] &= \sum_{b=1}^4 \alpha_{ab}^{+i} (G_{n+r}^b - 2(1 - \gamma)nQ_{n+r}^b) \\ [A_n^{-i}, G_r^a] &= \sum_{b=1}^4 \alpha_{ab}^{-i} (G_{n+r}^b + 2\gamma nQ_{n+r}^b) \\ [A_n^{\pm i}, A_m^{\pm j}] &= \sum_{k=1}^3 \epsilon^{ijk} A_{n+m}^{\pm k} - \frac{k^{\pm}}{2} n\delta^{ij}\delta_{n+m,0} \\ [A_n^{\pm i}, Q_r^a] &= \sum_{b=1}^4 \alpha_{ab}^{\pm i} Q_{n+r}^b \\ [U_n, G_r^a] &= nQ_{n+r}^a \\ [U_n, U_m] &= -\frac{c}{12\gamma(1 - \gamma)} n\delta_{n+m,0} \\ \{Q_r^a, G_s^b\} &= 2 \sum_{i=1}^3 (\alpha_{ab}^{+i}A_{r+s}^{+i} - \alpha_{ab}^{-i}A_{r+s}^{-i}) + \delta^{ab}U_{r+s} \\ \{Q_r^a, Q_s^b\} &= -\frac{c}{12\gamma(1 - \gamma)} \delta^{ab}\delta_{r+s,0} \\ 0 &= [A_n^{+i}, A_m^{-j}] = [U_n, Q_r^a] = [U_n, A_m^{\pm i}]. \end{aligned} \tag{6}$$

Φ is any of the generators G, A, U or Q .

3. Non-reductive $N = 4$ superconformal algebra

Let us introduce the linearly combined generators

$$\begin{aligned} G^{\pm\alpha} &= \frac{1}{2}(G^1 \pm iG^2) & G^{\pm\beta} &= \frac{1}{2}(G^3 \pm iG^4) \\ Q^{\pm\alpha} &= \frac{1}{2}(Q^1 \pm iQ^2) & Q^{\pm\beta} &= \frac{1}{2}(Q^3 \pm iQ^4) \end{aligned} \tag{7}$$

and

$$E^{\pm} = A^{\pm 2} - iA^{\pm 1} \quad H^{\pm} = 2iA^{\pm 3} \quad F^{\pm} = -A^{\pm 2} - iA^{\pm 1}. \tag{8}$$

We immediately rescale and rename some of the generators

$$\begin{aligned} \phi^{-\alpha} &= 2\gamma Q^{-\alpha} & \phi^{-\beta} &= 2\gamma Q^{-\beta} \\ \phi^{\alpha} &= 2Q^{\alpha} & \phi^{\beta} &= 2Q^{\beta} \\ J &= E^+ & V &= \gamma H^+ & R &= \gamma F^+ \\ E &= E^- & H &= H^- & F &= F^-. \end{aligned} \tag{9}$$

Note that E^+ is not scaled. We also replace U with the spin-1 generator W :

$$W = U - \frac{1}{2}H^+ \tag{10}$$

whereby the generator briefly known as H^+ appears scaled as well as unscaled. The remaining five generators L and G are left unscaled. The conformal weights are ‘inherited’ by the new generators (7) to (10): $\Delta(\Phi) \in \{3/2, 1, \dots, 1, 1/2\}$ for $\Phi \in \{G^{\pm\alpha, \beta}, E, H, F, J, V, R, W, \phi^{\pm\alpha, \beta}\}$.

In terms of this equivalent set of generators, the non-vanishing (anti-)commutators of the large $N = 4$ SCA are

$$\begin{aligned}
[L_n, L_m] &= (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \\
[L_n, \Phi_r] &= ((\Delta(\Phi) - 1)n - r)\Phi_{n+r} \\
\{G_r^\alpha, G_s^{-\alpha}\} &= L_{r+s} + (r - s)\left(\frac{1}{2}V_{r+s} + \frac{1}{2}(1 - \gamma)H_{r+s}\right) + \frac{1}{6}c(r^2 - \frac{1}{4})\delta_{r+s,0} \\
\{G_r^\beta, G_s^{-\beta}\} &= L_{r+s} + (r - s)\left(\frac{1}{2}V_{r+s} - \frac{1}{2}(1 - \gamma)H_{r+s}\right) + \frac{1}{6}c(r^2 - \frac{1}{4})\delta_{r+s,0} \\
\{G_r^\alpha, G_s^\beta\} &= (r - s)\gamma J_{r+s} & \{G_r^{-\alpha}, G_s^{-\beta}\} &= -(r - s)R_{r+s} \\
\{G_r^\alpha, G_s^{-\beta}\} &= (r - s)(1 - \gamma)E_{r+s} & \{G_r^\beta, G_s^{-\alpha}\} &= (r - s)(1 - \gamma)F_{r+s} \\
[E_n, G_r^{-\alpha}] &= -G_{n+r}^{-\beta} - n\phi_{n+r}^{-\beta} & [E_n, G_r^\beta] &= G_{n+r}^\alpha + \gamma n\phi_{n+r}^\alpha \\
[H_n, G_r^\alpha] &= G_{n+r}^\alpha + \gamma n\phi_{n+r}^\alpha & [H_n, G_r^{-\alpha}] &= -G_{n+r}^{-\alpha} - n\phi_{n+r}^{-\alpha} \\
[H_n, G_r^\beta] &= -G_{n+r}^\beta - \gamma n\phi_{n+r}^\beta & [H_n, G_r^{-\beta}] &= G_{n+r}^{-\beta} + n\phi_{n+r}^{-\beta} \\
[F_n, G_r^\alpha] &= G_{n+r}^\beta + \gamma n\phi_{n+r}^\beta & [F_n, G_r^{-\beta}] &= -G_{n+r}^{-\alpha} - n\phi_{n+r}^{-\alpha} \\
[H_n, E_m] &= 2E_{n+m} & [H_n, F_m] &= -2F_{n+m} \\
[H_n, H_m] &= \frac{c}{3(1 - \gamma)}n\delta_{n+m,0} & [E_n, F_m] &= H_{n+m} + \frac{c}{6}n\delta_{n+m,0} \\
[E_n, \phi_r^{-\alpha}] &= -\phi_{n+r}^{-\beta} & [E_n, \phi_r^\beta] &= \phi_{n+r}^\alpha \\
[H_n, \phi_r^\alpha] &= \phi_{n+r}^\alpha & [H_n, \phi_r^{-\alpha}] &= -\phi_{n+r}^{-\alpha} \\
[H_n, \phi_r^\beta] &= -\phi_{n+r}^\beta & [H_n, \phi_r^{-\beta}] &= \phi_{n+r}^{-\beta} \\
[F_n, \phi_r^\alpha] &= \phi_{n+r}^\beta & [F_n, \phi_r^{-\beta}] &= -\phi_{n+r}^{-\alpha} \\
[J_n, G_r^{-\alpha}] &= -G_{n+r}^\beta + (1 - \gamma)n\phi_{n+r}^\beta & [J_n, G_r^{-\beta}] &= G_{n+r}^\alpha - (1 - \gamma)n\phi_{n+r}^\alpha \\
[V_n, G_r^\alpha] &= \gamma G_{n+r}^\alpha - \gamma(1 - \gamma)n\phi_{n+r}^\alpha & [V_n, G_r^{-\alpha}] &= -\gamma G_{n+r}^{-\alpha} + (1 - \gamma)n\phi_{n+r}^{-\alpha} \\
[V_n, G_r^\beta] &= \gamma G_{n+r}^\beta - \gamma(1 - \gamma)n\phi_{n+r}^\beta & [V_n, G_r^{-\beta}] &= -\gamma G_{n+r}^{-\beta} + (1 - \gamma)n\phi_{n+r}^{-\beta} \\
[R_n, G_r^\alpha] &= \gamma G_{n+r}^{-\beta} - (1 - \gamma)n\phi_{n+r}^{-\beta} & [R_n, G_r^\beta] &= -\gamma G_{n+r}^{-\alpha} + (1 - \gamma)n\phi_{n+r}^{-\alpha} \\
[V_n, J_m] &= 2\gamma J_{n+m} & [V_n, R_m] &= -2\gamma R_{n+m} \\
[V_n, V_m] &= \frac{c}{3}\gamma n\delta_{n+m,0} & [J_n, R_m] &= V_{n+m} + \frac{c}{6}n\delta_{n+m,0} \\
[J_n, \phi_r^{-\alpha}] &= -\gamma\phi_{n+r}^\beta & [J_n, \phi_r^{-\beta}] &= \gamma\phi_{n+r}^\alpha \\
[V_n, \phi_r^\alpha] &= \gamma\phi_{n+r}^\alpha & [V_n, \phi_r^{-\alpha}] &= -\gamma\phi_{n+r}^{-\alpha} \\
[V_n, \phi_r^\beta] &= \gamma\phi_{n+r}^\beta & [V_n, \phi_r^{-\beta}] &= -\gamma\phi_{n+r}^{-\beta} \\
[R_n, \phi_r^\alpha] &= \phi_{n+r}^{-\beta} & [R_n, \phi_r^\beta] &= -\phi_{n+r}^{-\alpha} \\
\{\phi_r^\alpha, G_s^{-\alpha}\} &= W_{r+s} + \frac{1}{2}H_{r+s} & \{\phi_r^\alpha, G_s^\beta\} &= -J_{r+s} \\
\{\phi_r^\alpha, G_s^{-\beta}\} &= E_{r+s} & \{\phi_r^\beta, G_s^\alpha\} &= J_{r+s} \\
\{\phi_r^\beta, G_s^{-\alpha}\} &= F_{r+s} & \{\phi_r^\beta, G_s^{-\beta}\} &= W_{r+s} - \frac{1}{2}H_{r+s}
\end{aligned} \tag{11}$$

$$\begin{aligned}
 \{\phi_r^{-\alpha}, G_s^\alpha\} &= V_{r+s} + \gamma W_{r+s} - \frac{1}{2}\gamma H_{r+s} & \{\phi_r^{-\alpha}, G_s^\beta\} &= -\gamma F_{r+s} \\
 \{\phi_r^{-\alpha}, G_s^{-\beta}\} &= R_{r+s} & \{\phi_r^{-\beta}, G_s^\alpha\} &= -\gamma E_{r+s} \\
 \{\phi_r^{-\beta}, G_s^{-\alpha}\} &= -R_{r+s} & \{\phi_r^{-\beta}, G_s^\beta\} &= V_{r+s} + \gamma W_{r+s} + \frac{1}{2}\gamma H_{r+s} \\
 \{\phi_r^\alpha, \phi_s^{-\alpha}\} &= -\frac{c}{6(1-\gamma)}\delta_{r+s,0} & \{\phi_r^\beta, \phi_s^{-\beta}\} &= -\frac{c}{6(1-\gamma)}\delta_{r+s,0} \\
 [W_n, G_r^\alpha] &= -\frac{1}{2}G_{n+r}^\alpha + (1-\gamma/2)n\phi_{n+r}^\alpha & [W_n, G_r^{-\alpha}] &= \frac{1}{2}G_{n+r}^{-\alpha} + \frac{1}{2}n\phi_{n+r}^{-\alpha} \\
 [W_n, G_r^\beta] &= -\frac{1}{2}G_{n+r}^\beta + (1-\gamma/2)n\phi_{n+r}^\beta & [W_n, G_r^{-\beta}] &= \frac{1}{2}G_{n+r}^{-\beta} + \frac{1}{2}n\phi_{n+r}^{-\beta} \\
 [W_n, J_m] &= -J_{n+m} & [W_n, V_m] &= -\frac{c}{6}n\delta_{n+m,0} & [W_n, R_m] &= R_{n+m} \\
 [W_n, \phi_r^\alpha] &= -\frac{1}{2}\phi_{n+r}^\alpha & [W_n, \phi_r^{-\alpha}] &= \frac{1}{2}\phi_{n+r}^{-\alpha} \\
 [W_n, \phi_r^\beta] &= -\frac{1}{2}\phi_{n+r}^\beta & [W_n, \phi_r^{-\beta}] &= \frac{1}{2}\phi_{n+r}^{-\beta} \\
 [W_n, W_m] &= -\frac{c}{12(1-\gamma)}n\delta_{n+m,0}.
 \end{aligned}$$

As usual, we let Φ denote any of the 15 generators different from L .

3.1. Inönü–Wigner contraction

The IW contraction we are interested in corresponds to considering the limit $\gamma \rightarrow 0$ of the large $N = 4$ SCA in the form (11). Even though the limit appears singular from the point of view of the redefinitions (9), the algebra (11) does not display any divergences. It merely reduces to

$$\begin{aligned}
 [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0} \\
 [L_n, \Phi_r] &= ((\Delta(\Phi) - 1)n - r)\Phi_{n+r} \\
 \{G_r^\alpha, G_s^{-\alpha}\} &= L_{r+s} + \frac{1}{2}(r-s)(V_{r+s} + H_{r+s}) + \frac{1}{6}c(r^2 - \frac{1}{4})\delta_{r+s,0} \\
 \{G_r^\beta, G_s^{-\beta}\} &= L_{r+s} + \frac{1}{2}(r-s)(V_{r+s} - H_{r+s}) + \frac{1}{6}c(r^2 - \frac{1}{4})\delta_{r+s,0} \\
 \{G_r^\alpha, G_s^{-\beta}\} &= (r-s)E_{r+s} & \{G_r^\beta, G_s^{-\alpha}\} &= (r-s)F_{r+s} \\
 \{G_r^{-\alpha}, G_s^{-\beta}\} &= -(r-s)R_{r+s} \\
 [E_n, G_r^{-\alpha}] &= -G_{n+r}^{-\beta} - n\phi_{n+r}^{-\beta} & [E_n, G_r^\beta] &= G_{n+r}^\alpha \\
 [H_n, G_r^\alpha] &= G_{n+r}^\alpha & [H_n, G_r^{-\alpha}] &= -G_{n+r}^{-\alpha} - n\phi_{n+r}^{-\alpha} \\
 [H_n, G_r^\beta] &= -G_{n+r}^\beta & [H_n, G_r^{-\beta}] &= G_{n+r}^{-\beta} + n\phi_{n+r}^{-\beta} \\
 [F_n, G_r^\alpha] &= G_{n+r}^\beta & [F_n, G_r^{-\beta}] &= -G_{n+r}^{-\alpha} - n\phi_{n+r}^{-\alpha} \\
 [H_n, E_m] &= 2E_{n+m} & [H_n, F_m] &= -2F_{n+m} \\
 [H_n, H_m] &= \frac{c}{3}n\delta_{n+m,0} & [E_n, F_m] &= H_{n+m} + \frac{c}{6}n\delta_{n+m,0} \\
 [E_n, \phi_r^{-\alpha}] &= -\phi_{n+r}^{-\beta} & [H_n, \phi_r^{-\alpha}] &= -\phi_{n+r}^{-\alpha} \\
 [H_n, \phi_r^{-\beta}] &= \phi_{n+r}^{-\beta} & [F_n, \phi_r^{-\beta}] &= -\phi_{n+r}^{-\alpha} \\
 [V_n, G_r^{-\alpha}] &= n\phi_{n+r}^{-\alpha} & [V_n, G_r^{-\beta}] &= n\phi_{n+r}^{-\beta} \\
 [R_n, G_r^\alpha] &= -n\phi_{n+r}^{-\beta} & [R_n, G_r^\beta] &= n\phi_{n+r}^{-\alpha} \\
 \{\phi_r^{-\alpha}, G_s^\alpha\} &= V_{r+s} & \{\phi_r^{-\alpha}, G_s^{-\beta}\} &= R_{r+s} \\
 \{\phi_r^{-\beta}, G_s^{-\alpha}\} &= -R_{r+s} & \{\phi_r^{-\beta}, G_s^\beta\} &= V_{r+s}
 \end{aligned} \tag{12}$$

and

$$\begin{aligned}
[E_n, \phi_r^\beta] &= \phi_{n+r}^\alpha & [H_n, \phi_r^\alpha] &= \phi_{n+r}^\alpha \\
[H_n, \phi_r^\beta] &= -\phi_{r+n}^\beta & [F_n, \phi_r^\alpha] &= \phi_{n+r}^\beta \\
[J_n, G_r^{-\alpha}] &= -G_{n+r}^\beta + n\phi_{n+r}^\beta & [J_n, G_r^{-\beta}] &= G_{n+r}^\alpha - n\phi_{n+r}^\alpha \\
[J_n, R_m] &= V_{n+m} + \frac{c}{6}n\delta_{n+m,0} & & \\
[R_n, \phi_r^\alpha] &= \phi_{n+r}^{-\beta} & [R_n, \phi_r^\beta] &= -\phi_{n+r}^{-\alpha} \\
\{\phi_r^\alpha, G_s^{-\alpha}\} &= W_{r+s} + \frac{1}{2}H_{r+s} & \{\phi_r^\alpha, G_s^\beta\} &= -J_{r+s} & \{\phi_r^\alpha, G_s^{-\beta}\} &= E_{r+s} \\
\{\phi_r^\beta, G_s^\alpha\} &= J_{r+s} & & & \{\phi_r^\beta, G_s^{-\alpha}\} &= F_{r+s} \\
\{\phi_r^\beta, G_s^{-\beta}\} &= W_{r+s} - \frac{1}{2}H_{r+s} & & & & \\
\{\phi_s^\alpha, \phi_s^{-\alpha}\} &= -\frac{c}{6}\delta_{r+s,0} & & & \{\phi_r^\beta, \phi_s^{-\beta}\} &= -\frac{c}{6}\delta_{r+s,0} \\
[W_n, G_r^\alpha] &= -\frac{1}{2}G_{n+r}^\alpha + n\phi_{n+r}^\alpha & [W_n, G_r^{-\alpha}] &= \frac{1}{2}G_{n+r}^{-\alpha} + \frac{1}{2}n\phi_{n+r}^{-\alpha} \\
[W_n, G_r^\beta] &= -\frac{1}{2}G_{n+r}^\beta + n\phi_{n+r}^\beta & [W_n, G_r^{-\beta}] &= \frac{1}{2}G_{n+r}^{-\beta} + \frac{1}{2}n\phi_{n+r}^{-\beta} \\
[W_n, J_m] &= -J_{n+m} & [W_n, V_m] &= -\frac{c}{6}n\delta_{n+m,0} & [W_n, R_m] &= R_{n+m} \\
[W_n, \phi_r^\alpha] &= -\frac{1}{2}\phi_{n+r}^\alpha & & & [W_n, \phi_r^{-\alpha}] &= \frac{1}{2}\phi_{n+r}^{-\alpha} \\
[W_n, \phi_r^\beta] &= -\frac{1}{2}\phi_{n+r}^\beta & & & [W_n, \phi_r^{-\beta}] &= \frac{1}{2}\phi_{n+r}^{-\beta} \\
[W_n, W_m] &= -\frac{c}{12}n\delta_{n+m,0} & & & &
\end{aligned} \tag{13}$$

This algebra is singly extended, parametrized by the central charge $c = 6k^-$. For reasons which will become clear shortly, we have written the many non-vanishing (anti-)commutators in two separate families: (12) and (13). Half-integer moding of the fermionic generators G and ϕ corresponds to a Neveu–Schwarz sector, while integer moding corresponds to a Ramond sector.

The Jacobi identities (2) are in general not ensured after an IW contraction. However, we have carried out explicitly the tedious job of verifying (2) for the 16 generators of (12) and (13). Hence, this new SCA is well-defined. It is asymmetric in the way the supercurrents are treated, since $\{G_r^{-\alpha}, G_s^{-\beta}\} = -(r-s)R_{r+s}$ while $\{G_r^\alpha, G_s^\beta\} = 0$. This interesting feature was also present in the SCA of [5].

We observe that (12) is a subalgebra of the full SCA. When compared to the asymmetric SCA of [5], this subalgebra is equivalent to the asymmetric one, provided the spin-1 generator R is considered the derivative of the spin-0 generator S of [5]

$$R_n = nS_n. \tag{14}$$

This means that the asymmetric SCA is slightly bigger than (12) since certain commutators involving S_0 are non-vanishing [5].

In the extension governed by (13), we cannot consider the spin-1 field R straightforwardly as the derivative of a spin-0 field S . This follows from the commutator $[J_n, R_m]$, for example, where the right-hand side involves the spin-1 field V .

Equivalent, non-reductive $N = 4$ SCAs may be constructed by similar IW contractions. First, scaling E^+ and Q^+ instead of F^+ and Q^- (9) will lead to an isomorphic SCA with the roles of G^+ and G^- interchanged. Similarly, one could consider the limit $\gamma \rightarrow 1$ in which case one would have to scale H^- and either F^- or E^- (and leave H^+ , E^+ and F^+ unscaled). In this limit the central extension becomes $c = 6k^+$.

4. Affine extensions of Lie algebras

Let us conclude by classifying the possible affine extensions of the four-dimensional and non-reductive Lie algebra g . It is generated by the zero-modes $\{J_0, V_0, R_0, W_0\}$ of the particular affine Lie algebra appearing in our construction above

$$\begin{aligned}
 [J_n, R_m] &= V_{n+m} + \frac{c}{6}n\delta_{n+m,0} \\
 [W_n, J_m] &= -J_{n+m} \quad [W_n, V_m] = -\frac{c}{6}n\delta_{n+m,0} \\
 [W_n, R_m] &= R_{n+m} \quad [W_n, W_m] = -\frac{c}{12}n\delta_{n+m,0} \\
 0 &= [J_n, J_m] = [V_n, J_m] = [V_n, V_m] = [V_n, R_m] = [R_n, R_m].
 \end{aligned}
 \tag{15}$$

Abbreviating the zero-modes by $\{J, V, R, W\}$, the non-vanishing commutators of the algebra g are

$$[J, R] = V \quad [W, J] = -J \quad [W, R] = R.
 \tag{16}$$

This algebra is recognized as the semi-direct sum

$$g = u(1) \oplus n_3
 \tag{17}$$

where the Lie algebra n_3 of strictly upper-triangular 3×3 -matrices is the biggest non-trivial ideal of g . The $u(1)$ is generated by $\{W\}$, while n_3 is generated by $\{J, V, R\}$. g is seen to be solvable.

An affine extension of the generic Lie algebra

$$[j_a, j_b] = f_{ab}{}^c j_c
 \tag{18}$$

is governed by the central extension (or level) k , and a symmetric, bilinear and invariant two-form κ :

$$\begin{aligned}
 \kappa_{ab} &= \kappa(j_a, j_b) \\
 \kappa([j_a, j_b], j_c) &= \kappa(j_a, [j_b, j_c]).
 \end{aligned}
 \tag{19}$$

The resulting affine Lie algebra reads

$$[j_{a,n}, j_{b,m}] = f_{ab}{}^c j_{c,n+m} + nk\kappa_{ab}\delta_{n+m,0}.
 \tag{20}$$

The invariance of κ , in particular, is required by the Jacobi identities of (20). Note that k may be absorbed in a rescaling of κ . However, it is convenient to factorize the extension into the Lie algebra-dependent object κ , and the purely affine entity k . We see that a classification of possible affine extensions of a given Lie algebra amounts to classifying the κ -forms of the Lie algebra.

The canonical choice of κ -form (19) is the ordinary Cartan–Killing form

$$\kappa_{ab} = f_{ac}{}^d f_{bd}{}^c.
 \tag{21}$$

For simple Lie algebras, it is unique up to an overall scaling (the normalization chosen here is unconventional but irrelevant to our purpose). For non-reductive Lie algebras, on the other hand, there will in general exist several inequivalent κ -forms. Here we shall classify them in the case of g (16).

The approach is straightforward and can be applied to any finite-dimensional Lie algebra. The defining properties of the κ -form imply the anti-symmetry

$$f_{cab} = -f_{cba} \quad \text{where} \quad f_{cab} = f_{ca}{}^d \kappa_{db}.
 \tag{22}$$

It should be noted that since κ may be degenerate, it in general cannot be used as a metric whose inverse can raise indices. Now, considering f_{cab} as a matrix element of the $d \times d$ -matrix

f_c (d is the dimension of the Lie algebra), the anti-symmetry (22) imposes constraints on κ . Along with the symmetry of κ , $\kappa_{ab} = \kappa_{ba}$, those are the only constraints to impose. Thus, the classification is achieved by an analysis of the d matrices f_c . In the case of g (16) we find the general κ -form

$$\kappa = \lambda \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \mu \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \bar{1} \\ 1 & 0 & 0 & 0 \\ 0 & \bar{1} & 0 & 0 \end{pmatrix}. \quad (23)$$

Here $\bar{1} \equiv -1$. The matrix elements are labelled according to the order $\{J, V, R, W\}$, i.e. $\kappa_{24} = \kappa(V, W)$, etc. We see that the ordinary Cartan–Killing form corresponds to $(\lambda, \mu) = (2, 0)$, while the affine extension (15) corresponds to $(\lambda, \mu) = (-1, 2)$. In the latter case, the level has been normalized to $k = c/12$.

As a final comment, we observe that in terms of the linear combinations

$$U_n^+ = V_n - 2W_n \quad U_n^- = -2W_n \quad (24)$$

equation (15) becomes

$$\begin{aligned} [J_n, R_m] &= U_{n+m}^+ - U_{n+m}^- + \frac{c}{6}n\delta_{n+m,0} \\ [U_n^\pm, J_m] &= 2J_{n+m} \quad [U_n^\pm, R_m] = -2R_{n+m} \\ [U_n^\pm, U_m^\pm] &= \pm \frac{c}{3}n\delta_{n+m,0} \\ 0 &= [U_n^+, U_m^-] = [J_n, J_m] = [R_n, R_m]. \end{aligned} \quad (25)$$

Thus, the affine Lie algebra (15) is seen to contain a level $k = c/6$ affine $su(2)$ Lie algebra generated by $\{J, U^+, R\}$, and a level $k = -c/6$ affine $su(2)$ Lie algebra generated by $\{-J, U^-, R\}$ (or equivalently by $\{J, U^-, -R\}$). We have used the term contain deliberately to emphasize that they are obviously not subalgebras.

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